كلية التقنية الالكترونية- طرابلس

القسم:الاتصالات التخصص:إتصالات عام الفصل:السادس المادة:نظرية المعلومات و الترميز

# CM303 Final Exam- Autumn 2014

**Answer All questions** 28/01/2015

Time allowed: TWO hours

**QUESTION 1: (20%)** a simple language consists of only two symbols A and B produced in a long continuous sequence. Find the single, joint and conditional probabilities of A and B, assuming that the values found from the limited sequence below are typical of a very long sequence ( assumed that the 2 1st letter is A in order to have 20 pairs of pairs of symbols ). Evaluate the conditional entropy for the sequence.

AABBBAAAABBAAABBBAAA

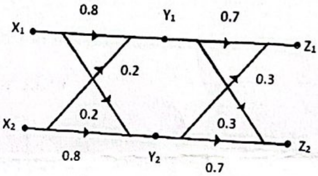
**QUESTION 2: (20%)** A channel has the following Channel matrix.

$$P(Y/X) = \begin{bmatrix} 1-P & P & 0 \\ 0 & P & 1-P \end{bmatrix}$$

(a)Draw the Channel diagram.

(b)If the source has equally likely outputs, Compute the pfobabilities associated with the channel outputs for P=0.2

**QUESTION 3: (20%)** Two BSC's are connected in cascade as shown in the figure:




a)Find the Channel Matrix of the resultant channel.

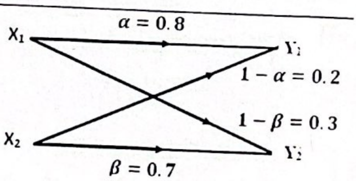
(b)Find $P(Z_1)$ and $P(Z_2)$, if $P(X_1)$ =0.6, $P(X_2)$ =0.4

**QUESTION 4: (20%)** Consider a binary symmetric communication channel, whose input source is the alphabet X = {0, 1} with probabilities {0.5, 0.5}; whose output alphabet is Y = {0, 1}; and whose channel matrix is

$$\begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}$$

1. What is the entropy of the source, H(X)?
2. What is the probability distribution of the outputs, p(Y), and the entropy of this output distribution, H(Y)?
3. What is the joint probability distribution for the source and the output, p(X,Y), and what is the joint entropy, H(X,Y)?
4. What is the mutual information of this channel, I(X; Y)?
5. How many values are there for $\epsilon$ for which the mutual information of this channel is maximal? What are those values, and what then is the capacity of such a channel in bits?
6. For what value of $\epsilon$ is the capacity of this channel minimal? What is the channel capacity in that case?

**QUESTION 5: (20%)**Find the Mutual Information for the channel shown in the figure. Given that $P(X_1)$ =0.6 and $P(X_2)$ =0.4.

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الإجابة النموذجية

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**QUESTION 1:**

**Solution:**

p(A) = 12/20, p(B)=8/20, by counting As and Bs
p(AA)= 9/20, p(BB) 5/20
p(AB)= 3/20, p(BA) 3/20, by counting pairs
p(A/B)=3/8, p(B/B)=5/8, by counting As or Bs after B
p(A/A) =9/12, p(B/A) = 3/12, by counting As or Bs after A
There are four pairs of symbols AA, BB, AB and BA. Therefore
H(j/i) = -[p(AA) log p(A/A) + p(BB) log (p(B/B)) + p(AB) log p(A/B) + p(BA) log (B/A)]
= 9/20 log 9/12 + 5/20 log 5/8 + 3/20 log 3/8 + 3/20 log 3/12
= 0.868 bit/symbol
if no intersymbol influence had been present the information would have been given by
= -(0.6 log 0.6 + 0.4 log 0.4)
= 0.971 bits/symbol

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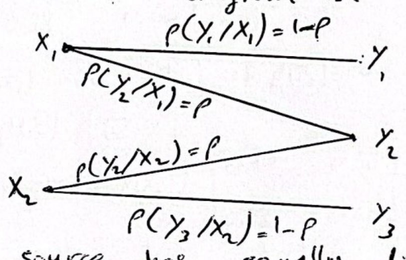
**QUESTION 2:**

**Solution:** A channel has the following matrix:

$$P(Y/X) = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix} = \begin{bmatrix} p(y_1/x_1) & p(y_2/x_1) & p(y_3/x_1) \\ p(y_1/x_2) & p(y_2/x_2) & p(y_3/x_2) \end{bmatrix}$$

(a) the channel diagram is:-




(b) if the source has equally likely outputs, Hence;
$p(x_1) = \frac{1}{2}$ & $p(x_2) = \frac{1}{2}$, total prob. = 1
and $p = 0.2$ (given)
the output probabilities are given as:-

$$\begin{bmatrix} p(y_1) \\ p(y_2) \\ p(y_3) \end{bmatrix} = [p(x_1) \quad p(x_2)] \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$\begin{bmatrix} P(y_1) \\ P(y_2) \\ P(y_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times 0.8 + 0 \\ \frac{1}{2} \times 0.2 + \frac{1}{2} \times 0.2 \\ \frac{1}{2} \times 0 + \frac{1}{2} \times 0.8 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}$$

$\therefore P(y_1) = 0.4$, $P(y_2) = 0.2$, $P(y_3) = 0.4$.

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**QUESTION 3:**
**Solution:**

ⓐ channel Matrix:-

$$P(Y/X) = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P(Z/Y) = \begin{bmatrix} P(z_1/y_1) & P(z_2/y_1) \\ P(z_1/y_2) & P(z_2/y_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Hence, the resultant channel Matrix is : $P(Z/X) = P(Y/X) \cdot P(Z/Y)$

$$P(Z/X) = \begin{bmatrix} P(z_1/x_1) & P(z_2/x_1) \\ P(z_1/x_2) & P(z_2/x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

ⓑ Find $P(z_1)$ and $P(z_2)$, if $P(x_1) = 0.6$ and $P(x_2) = 0.4$

$$P(Z) = P(X)\, P(Z/X)$$

$$= [P(x_1) \quad P(x_2)] \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= [0.6 \quad 0.4] \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.372 + 0.152 \\ 0.228 + 0.248 \end{bmatrix}$$

$$p(Z) = \begin{bmatrix} 0.524 \\ 0.476 \end{bmatrix}$$

thus :- $p(Z_1) = 0.524$

$p(Z_2) = 0.476$

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**QUESTION 4:**

**Solution:**

① Entropy of the source is:- $H(X) = \sum_{i=1}^{2} P_i \log_2 \left( \frac{1}{P_i} \right)$

$H(X) = \frac{1}{2} \log_2 2 + \frac{1}{2} \log 2 = 1$ bits/symbol

② output prob. are :-

$P(y=0) = (0.5)(1-\epsilon) + 0.5\epsilon = 0.5$

$P(y=1) = (0.5)(1-\epsilon) + (0.5)\epsilon = 0.5$

∴ Entropy of output distribution is:-

$H(Y) = \frac{1}{2} \log 2 + \frac{1}{2} \log 2 = 1$ bits/symbol

input distribution $H(X)$ = output distribution $H(Y)$

③ Joint probability distribution $p(X, Y)$ is

$$\begin{bmatrix} 0.5(1-\epsilon) & 0.5\epsilon \\ 0.5\epsilon & 0.5(1-\epsilon) \end{bmatrix}$$

and Joint entropy $H(X, Y) = -\sum_{x,y} p(x,y) \log_2 (p(x,y))$

$\left[ \text{(i.e)} \sum_{i=1}^{M} \sum_{j=1}^{M} p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} \Leftrightarrow \sum_x \sum_y p(x,y) \log_2 \frac{1}{p(x,y)} \right]$

$= -(1-\epsilon) \log(0.5(1-\epsilon)) - \epsilon \log(0.5\epsilon) = (1-\epsilon) - (1-\epsilon)\log(1-\epsilon) + \epsilon - \epsilon \log(\epsilon)$

$= 1 - \epsilon \log(\epsilon) - (1-\epsilon) \log(1-\epsilon)$

④ the mutual information is $I(X;Y) = H(X) + H(Y) - H(X, Y)$, from above we can calculate $I(X;Y) = 1 + \epsilon \log(\epsilon) + (1-\epsilon) \log(1-\epsilon)$

⑤ in the two cases of $\epsilon = 0$, & $\epsilon = 1$ (perfect transmission & perfectly erroneous transmission), the mutual information reaches it's maximum of 1 bit & this is also the channel capacity.

⑤ if $\epsilon = 0.5$, the channel capacity is minimal and equal to 0.

**QUESTION 5:**

**Solution:** $p(x_1) = 0.6$ , $p(x_2) = 0.4$ ,

let $p(x_1) = p = 0.6$ , $p(x_2) = 1-p = 1-0.6 = 0.4$

entropy of the source is:-

① $H(X) = p\left(\log_2 \frac{1}{p}\right) + (1-p)\log_2 \frac{1}{(1-p)} = 0.6\log_2\frac{1}{0.6} + 0.4\log_2\frac{1}{0.}$

$= 0.442 + 0.5287 = 0.9709$ bits/symbol

② to obtain $H(Y)$

~~output~~ probabilities of output symbols :

$$\begin{bmatrix} p(y_1) \\ p(y_2) \end{bmatrix} = [p(x_1) \quad p(x_2)] \begin{bmatrix} p(y_1/x_1) & p(y_2/x_1) \\ p(y_1/x_2) & p(y_2/x_2) \end{bmatrix}$$

$$= [0.6 \quad 0.4] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.48 + 0.12 \\ 0.12 + 0.28 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$\therefore H(Y) = 0.6\log_2\frac{1}{0.6} + 0.4\log_2\frac{1}{0.4} = 0.442 + 0.5287$ ~~bits/symbol~~

$= 0.9709$ bits/symbol.

thus $\boxed{H(X) = H(Y)}$

③ to obtain $H(Y/X)$

$$H(Y/X) = \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\log_2\frac{1}{p(y_j/x_i)}$$

As we know :-

$p(x_i, y_j) = p(y_j/x_i)\, p(x_i)$

$p(x_1, y_1) = p(y_1/x_1)\, p(x_1) = \alpha p$

$p(x_1, y_2) = p(y_2/x_1)\, p(x_1) = (1-\alpha)p$

$p(x_2, y_1) = p(y_1/x_2)\, p(x_2) = (1-\beta)(1-p)$

$p(x_2, y_2) = p(y_2/x_2)\, p(x_2) = \beta(1-p)$

$$H(Y/X) = p(x_1, y_1)\log_2\frac{1}{p(y_1/x_1)} + p(x_1, y_2)\log_2\frac{1}{p(y_2/x_1)}$$
$$+ p(x_2, y_1)\log_2\frac{1}{p(y_1/x_2)} + p(x_2, y_2)\log_2\frac{1}{p(y_2/x_2)}$$

$$= \alpha p \log_2 \frac{1}{\alpha} + (1-\alpha)p \cancel{\log_2} \log_2 \frac{1}{1-\alpha} + (1+\beta)(1-p)$$

$$\log_2 \frac{1}{1-\beta} + \beta(1-p) \log_2 \frac{1}{\beta}$$

$$H(Y/X) = (0.8)(0.6) \log_2 \frac{1}{0.8} + (0.2)(0.6) \log_2 \frac{1}{0.2} + (0.3)(0.4) \log_2 \frac{1}{0.3}$$

$$+ (0.7)(0.4) \log_2 \frac{1}{0.7} = 0.1545 + 0.2786 + 0.2084 + 0.1441 =$$

$= 0.7855$ bits/symbol

(4) to obtain mutual information:

$$I(Y;X) = H(Y) - H(Y/X) = 0.9709 - 0.7855 =$$

$= 0.1853$ bits/symbol.

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